

DR. JIM'S MATHSTRAVAGANZA VOL.1 – Infinite sums, products and limits

This program contains seven convergent infinite sums, one infinite product, one limiting value, and one combination of sum and limit, in which the running value of the evaluated expression is shown on screen as the terms increase. If a sum converges slowly, after a large number of terms, the Spectrum will eventually not be able to handle the ever-decreasing value to be added and will effectively be adding 0 (or, in the case of an infinite product, multiplying by 1), thus the target shown on screen will never be reached, but still shows the sum approaching the target. For formal proofs of all of these, read an advanced maths book.

A. Gregory-Leibniz formula for π (converges extremely slowly):

July 2012

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots = \frac{\pi}{4}$$

B. Basel Problem (as solved by Leonhard Euler):

December 2006

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \dots = \frac{\pi^2}{6}$$

C. Wallis product for π :

February 2022

$$\prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = \prod_{k=1}^{\infty} \left(\frac{2k}{2k-1} \times \frac{2k}{2k+1} \right) = \left(\frac{2}{1} \times \frac{2}{3} \right) \times \left(\frac{4}{3} \times \frac{4}{5} \right) \times \left(\frac{6}{5} \times \frac{6}{7} \right) \times \left(\frac{8}{7} \times \frac{8}{9} \right) \dots = \frac{\pi}{2}$$

D. Nilakantha series for π :

February 2022

$$3 + \sum_{k=0}^{\infty} \frac{4(-1)^k}{(2k+2)(2k+3)(2k+4)} = 3 + \frac{4}{2 \times 3 \times 4} - \frac{4}{4 \times 5 \times 6} + \frac{4}{6 \times 7 \times 8} - \frac{4}{8 \times 9 \times 10} + \frac{4}{10 \times 11 \times 12} \dots = \pi$$

E. Madhava of Sangamagrama's improved infinite sum for π

(converges to 11 dp in 21 terms, so is very rapid):

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$$\sum_{k=0}^{\infty} \sqrt{12} \left(\frac{(-1)^k}{3^k(2k+1)} \right) = \sqrt{12} \left(\frac{1}{1 \times 3^0} - \frac{1}{3 \times 3^1} + \frac{1}{5 \times 3^2} - \frac{1}{7 \times 3^3} + \frac{1}{9 \times 3^4} \dots \right) = \pi$$

F. Bailey-Borwein-Plouffe formula for π

July 2012

(converges to the Spectrum's limit in 5 terms and stops with 6 Number too big at 32 terms):

$$\sum_{k=0}^{\infty} \left[\frac{1}{16^k} \left(\frac{4}{8k+1} \right) - \left(\frac{2}{8k+4} \right) - \left(\frac{1}{8k+5} \right) - \left(\frac{1}{8k+6} \right) \right] = \pi$$

G. Infinite limit of continuously compounded interest calculates e :

February 2022

$$e = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k$$

H. An infinite sum to produce e involves factorials - these can be handled without a lookup table, although the denominator will soon cause the program to stop with 6 Number too big.

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$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \dots = e^x, \text{ hence } \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \dots = e$$

I. Euler-Mascheroni constant:

May 2019

$$\gamma = \lim_{x \rightarrow \infty} \left(\sum_{k=1}^x \frac{1}{k} - \ln x \right)$$

J. Mercator series:

February 2022

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots = \ln(1+x), \text{ hence } \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots = \ln 2$$